

10.3 More on Limits

Name: _____

(Not to be confused with moron limits. Whatever those are.)

Objectives: Students will be able to use the properties of limits and evaluate one-sided limits, two-sided limits and limits involving infinity.

When we write $\lim_{x \rightarrow a} f(x) = L$, we mean that $f(x)$ gets arbitrarily

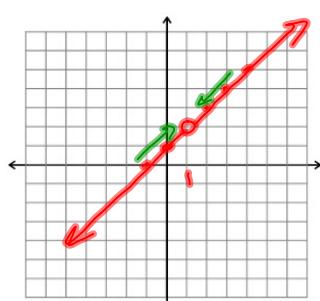
close to L as x gets arbitrarily close (but not equal) to a .



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Let's find $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = \lim_{x \rightarrow 1} x+1$ (HOLE at $x=1$)

Graphically



$$\lim_{x \rightarrow 1} x+1 = 2$$

Numerically

x	f(x)
.9	1.9
.99	1.99
.999	1.999
1.001	2.001 > 2
1.01	2.01
1.1	2.1

Algebraically

$$\begin{aligned} \lim_{x \rightarrow 1} (x+1) \\ = 1+1 \\ = 2 \end{aligned}$$

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Notation:

$\lim_{x \rightarrow c^-} f(x)$: The limit as x approaches c from the left.

$\lim_{x \rightarrow c^+} f(x)$: The limit as x approaches c from the right.

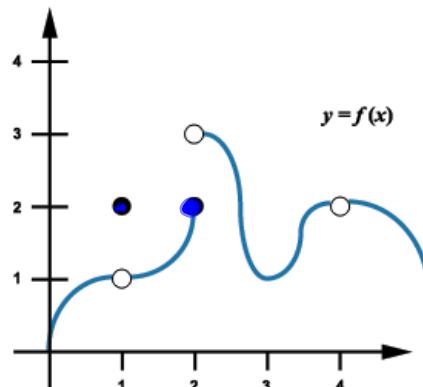
A function $f(x)$ has a limit as x approaches c if and only if the left-hand and right-hand limits at c exist and are equal. That is,

$\lim_{x \rightarrow c} f(x) = L$ if and only if $\lim_{x \rightarrow c^-} f(x) = L$ and $\lim_{x \rightarrow c^+} f(x) = L$.

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Examples Use the graph to find the following limits.

1.) $\lim_{x \rightarrow 1^-} f(x) = \underline{1}$



2.) $\lim_{x \rightarrow 1^+} f(x) = \underline{1}$

3.) $\lim_{x \rightarrow 1} f(x) = \underline{1}$

4.) $f(1) = \underline{2}$

5.) $\lim_{x \rightarrow 2^-} f(x) = \underline{2}$

6.) $\lim_{x \rightarrow 2^+} f(x) = \underline{3}$

7.) $\lim_{x \rightarrow 2} f(x) = \underline{\text{DNE}}$

8.) $f(2) = \underline{2}$

9.) $\lim_{x \rightarrow 4^-} f(x) = \underline{2}$

10.) $\lim_{x \rightarrow 4^+} f(x) = \underline{2}$

11.) $\lim_{x \rightarrow 4} f(x) = \underline{2}$

12.) $f(4) = \underline{\text{DNE}}$

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The easiest way to find a limit is by direct substitution.

Examples Find the limit by direct substitution if it exists.

$$1.) \lim_{x \rightarrow 3} (x - 1)^{12}$$

$$= (3-1)^{12}$$

$$= 2^{12}$$

$$= \boxed{4096}$$

$$2.) \lim_{x \rightarrow \pi} \ln(\sin(x/2))$$

$$= \ln(\sin \frac{\pi}{2})$$

$$= \ln 1$$

$$= \boxed{0}$$

$$\sim e^x = 1$$

Examples a.) Explain why you cannot use direct substitution to find the limit and b.) find the limit algebraically if it exists.

$$1.) \lim_{x \rightarrow 2} \frac{x^3 - 2x^2 + x - 2}{x - 2}$$

a.) Division by 0.

$$\lim_{x \rightarrow 2} \frac{x^2(x-2) + 1(x-2)}{x-2}$$

$$\lim_{x \rightarrow 2} \frac{(x-2)(x^2+1)}{(x-2)}$$

$$\lim_{x \rightarrow 2} (x^2+1)$$

$$= 2^2 + 1$$

$$= \boxed{5}$$

$$2.) \lim_{x \rightarrow 0} \frac{x-2}{x^2}$$

a.) Division by 0.

b.) Algebraically, there's nothing we can do. \therefore look at the graph.

$$\lim_{x \rightarrow 0} \frac{x-2}{x^2} = \text{DNE}$$

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Properties of Limits

If $\lim_{x \rightarrow c} f(x)$ and $\lim_{x \rightarrow c} g(x)$ both exist, then:

$$1.) \text{Sum Rule: } \lim_{x \rightarrow c} (f(x) + g(x)) = \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x)$$

$$2.) \text{Difference Rule: } \lim_{x \rightarrow c} (f(x) - g(x)) = \lim_{x \rightarrow c} f(x) - \lim_{x \rightarrow c} g(x)$$

$$3.) \text{Product Rule: } \lim_{x \rightarrow c} [f(x)g(x)] = \lim_{x \rightarrow c} f(x) \cdot \lim_{x \rightarrow c} g(x)$$

$$4.) \text{Constant Multiple Rule: } \lim_{x \rightarrow c} [kf(x)] = k \lim_{x \rightarrow c} f(x)$$

$$5.) \text{Quotient Rule: } \lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)} \quad > \quad \lim_{x \rightarrow c} g(x) \neq 0$$

$$6.) \text{Power Rule: } \lim_{x \rightarrow c} [f(x)^n] = (\lim_{x \rightarrow c} f(x))^n$$

$$7.) \text{Root Rule: } \lim_{x \rightarrow c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow c} f(x)}$$

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Example Use a table of values and a graph to find $\lim_{x \rightarrow 0} \frac{\sin x}{x}$.

$$x = 1$$

Examples Use the fact that $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ and the limit properties to find the following limits.

$$1.) \lim_{x \rightarrow 0} \frac{x + \sin x}{x}$$

$$\lim_{x \rightarrow 0} \left[1 + \frac{\sin x}{x} \right]$$

$$\lim_{x \rightarrow 0} 1 + \lim_{x \rightarrow 0} \frac{\sin x}{x}$$

$$1 + 1 = 2$$



$$2.) \lim_{x \rightarrow 0} \frac{\sin 3x}{x} = \lim_{x \rightarrow 0} \frac{\sin(x+2x)}{x}$$

$$= \lim_{x \rightarrow 0} \frac{\sin x \cos 2x + \sin 2x \cos x}{x}$$

$$= \lim_{x \rightarrow 0} \frac{\sin x(1 - 2\sin^2 x) + 2\sin x \cos^2 x}{x}$$

$$= \lim_{x \rightarrow 0} \frac{\sin x - 2\sin x \sin^2 x + 2\sin x \cos^2 x}{x}$$

$$= \lim_{x \rightarrow 0} \left[\frac{\sin x}{x} - \frac{2\sin x \sin^2 x}{x} + \frac{2\sin x \cos^2 x}{x} \right]$$

$$= \lim_{x \rightarrow 0} [1 - 2\sin^2 x + 2\cos^2 x]$$

$$= 1 + 2$$

$$= 3$$

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x	f(x)
- .1	.998
- .01	.999
- .001	1
.001	1
.01	.999
.1	.998

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Example Assume that $\lim_{x \rightarrow a} f(x) = 2$ and $\lim_{x \rightarrow a} g(x) = -3$. Find the limit.

1.) $\lim_{x \rightarrow a} (f(x) + g(x))$

$2 + (-3)$
-1

2.) $\lim_{x \rightarrow a} (3g(x) + 1)$

$3(-3) + 1 = -8$

3.) $\lim_{x \rightarrow a} (f(x)g(x))$

$2(-3)$
-6

4.) $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ $= \frac{2}{-3}$

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Limits at Infinity

Examples

1.) Find $\lim_{x \rightarrow \infty} \frac{x}{1 + 2^x}$ and $\lim_{x \rightarrow -\infty} \frac{x}{1 + 2^x}$.

2.) Find $\lim_{x \rightarrow \infty} e^{-x} + \sin x$ and $\lim_{x \rightarrow -\infty} e^{-x} + \sin x$.

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Does $\lim_{x \rightarrow a} f(x)$ exist? If it does, give its value. If it does not exist, give an explanation.

1.) $a = 1, f(x) = 2 - x, x < 1$
 $x + 1, x \geq 1$

2.) $a = -3, f(x) = 1 + x^2, x \geq -3$
 $8 - x, x < -3$

Homework: Pages 822-824: #1- 71 odd

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10.3 ICE

Name: _____

1.) Let $f(x) = x^3 + 8$ and $g(x) = x^2 - 2x + 4$.
 $x + 2$

a.) Explain why $f(x)$ and $g(x)$ are different functions.

b.) Explain why $\lim_{x \rightarrow -2} f(x) = \lim_{x \rightarrow -2} g(x)$.

2.) Let $f(x) = e^x$. Find each limit.

a.) $\lim_{x \rightarrow \infty} f(x)$ b.) $\lim_{x \rightarrow -\infty} f(x)$ c.) $\lim_{x \rightarrow 0} f(x)$

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3.) Let $f(x) = \frac{x^2 + 9}{x - 3}$ and $g(x) = \frac{x^2 - 9}{x - 3}$.

$$x - 3 \qquad \qquad x - 3$$

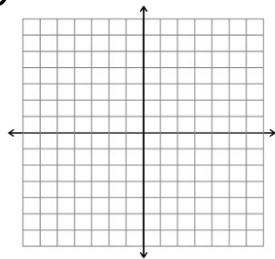
a.) Which function has a vertical asymptote at $x = 3$?

b.) Which limit exists: $\lim_{x \rightarrow 3} f(x)$ or $\lim_{x \rightarrow 3} g(x)$?

4.) Sketch a graph of a function $f(x)$ that satisfies the following conditions.

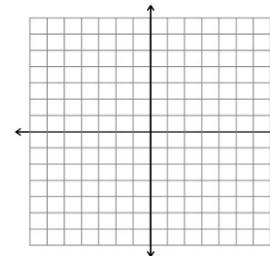
a.) $\lim_{x \rightarrow 3^-} f(x) = \infty$

$\lim_{x \rightarrow 3^+} f(x) = -\infty$



b.) $\lim_{x \rightarrow 4} f(x) = -\infty$, $\lim_{x \rightarrow \infty} f(x) = -\infty$

$\lim_{x \rightarrow -\infty} f(x) = 2$



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