Notes: Sequences and Series

A **Sequence** is a function whose DOMAIN is a set of consecutive integers. If a domain is NOT SPECIFIED it is understood that the domain starts with **1**. The values in the RANGE are called the **terms** of the sequence.

Domain	1	2	3	...	...	...	n
Range	a_1	a_2	a_3				a_n

Instead of using function notation, sequences are usually written using subscript notation.

Write the first five terms of the sequence.

1. $a_n = 2n+1$

$a_1 = 2(1)+1 = 3$
$a_2 = 2(2)+1 = 5$
$a_3 = 2(3)+1 = 7$
$a_4 = 2(4)+1 = 9$
$a_5 = 2(5)+1 = 11$
3, 5, 7, 9, 11

Write the first five terms of the sequence.

2. $a_n = 2-(-1)^n$

$a_1 = 2-(-1)^1 = 3$
$a_2 = 2-(-1)^2 = 1$
$a_3 = 2-(-1)^3 = 3$
$a_4 = 2-(-1)^4 = 1$
$a_5 = 2-(-1)^5 = 3$
3, 1, 3, 1, 3

Find the 3rd, 4th and 5th term of the sequence.

3. $a_n = \frac{2+(-1)^n}{n}$

$a_3 = \frac{2+(-1)^3}{3} = \frac{1}{3}$
$a_4 = \frac{2+(-1)^4}{4} = \frac{3}{4}$
$a_5 = \frac{2+(-1)^5}{5} = \frac{1}{5}$

If n is a positive integer, n **factorial** is defined as: $n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$.

As a special case, zero factorial is defined as: $0! = 1$.

1. Evaluate.

$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$= 5040$

2. Simplify the factorial expression.

$\frac{9!}{3!7!} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$
$= \frac{9 \cdot 8}{6} = 12$ OR $\frac{9 \cdot 8 \cdot 7!}{3! 7!}$

3. Write the first 5 terms of the sequence.

$a_n = \frac{2^n}{n!}$

$a_1 = \frac{2^1}{1!} = 2$
$a_2 = \frac{2^2}{2!} = 2$
$a_3 = \frac{2^3}{3!} = \frac{4}{3}$
$a_4 = \frac{2^4}{4!} = \frac{2}{3}$
$a_5 = \frac{2^5}{5!} = \frac{4}{15}$

4. Simplify the factorial expression.

$\frac{(n+1)!}{n!} = \frac{(n+1) \cdot (n) \cdot (n-1) \cdot (n-2) \cdot \ldots}{(n) \cdot (n-1) \cdot (n-2) \cdot \ldots}$
$n+1$
OR $\frac{(n+1) \cdot n!}{n!}$

A **Series** is the sum of the terms in a sequence. A series can be written with **Summation notation** where the sum of the first n terms of a sequence is represented by $\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots + a_n$, where i is called the **index of summation**.

n is the **upper limit** and 1 is the **lower limit**.

Find the sum.

5. $\sum_{i=1}^{4} (4i+1)$

$4(1)+1 = 5$
$4(2)+1 = 9$
$(4(3))+1 = 13$
$4(4)+1 = 17$
$5+9+13+17 = 44$

How many terms are in this series? 4

6. $\sum_{k=2}^{5} (2+k^3)$

$2+(2)^3 = 10$
$2+(3)^3 = 29$
$2+(4)^3 = 66$
$2+(5)^3 = 127$
Sum = 232

How many terms are in this series? 4

7. $\sum_{n=0}^{8} \left(\frac{1}{n!}\right)$

$\frac{1}{0!} = 1$ $\frac{1}{1!} = 1$ $\frac{1}{2!} = \frac{1}{2}$
$\frac{1}{3!} = \frac{1}{6}$ $\frac{1}{4!} = \frac{1}{24}$ $\frac{1}{5!} = \frac{1}{120}$
$\frac{1}{6!} = \frac{1}{720}$ $\frac{1}{7!} = \frac{1}{5040}$ $\frac{1}{8!} = \frac{1}{40320}$

How many terms are in this series? 9

To find the number of terms in a series: (upper − lower) + 1

Sum = 2.718

A **finite series** is the sum of the first n terms of the sequence, which is also called a **Partial sum**.

An **infinite series** is the sum of all the terms of the sequence.

Find the sum.

8. $\sum_{k=1}^{3} \left(\frac{3}{10^k}\right)$

$\frac{3}{10^1} + \frac{3}{10^2} + \frac{3}{10^3} = \frac{333}{1000}$
$.3 + .03 + .003 = .333$

*third partial sum

9. $\sum_{k=1}^{\infty} \left(\frac{3}{10^k}\right)$

$\frac{3}{10^1} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \frac{3}{10^5} + \ldots$
$= .3 + .03 + .003 + .0003 + .00003 \ldots$
$= .\overline{333} = \frac{1}{3}$

Arithmetic Sequences, also known as a discrete linear function, is a sequence for which consecutive terms have a **common difference**, d.

Determine whether or not the sequence is arithmetic. If it is, find the common difference.

1. 5, 8, 11, 14, 17,... (+3 +3 +3 +3) arithmetic, $d = 3$

2. 1, 4, 9, 16, 25,... (3 5 7 9) not arithmetic

3. $1, \frac{7}{6}, \frac{4}{3}, \frac{3}{2}, \frac{5}{3}, \ldots$ $d = \frac{1}{6}$; $\frac{6}{6}, \frac{7}{6}, \frac{8}{6}, \frac{9}{6}, \frac{10}{6}$ arithmetic

4. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ not arithmetic; $\frac{12}{12}, \frac{6}{12}, \frac{4}{12}, \frac{3}{12}$

Writing an explicit formula/rule for an arithmetic sequence a_n.

Fill in the missing terms from the sequence:

n	1	2	3	4	5	6	7	8
a_n	4	7	10	13	16	19	22	25

$d = 3$

Expanded:

$a_2 = 4 + 3$

$a_3 = 4 + 3 + 3$

$a_4 = 4 + 3 + 3 + 3$

Condensed:

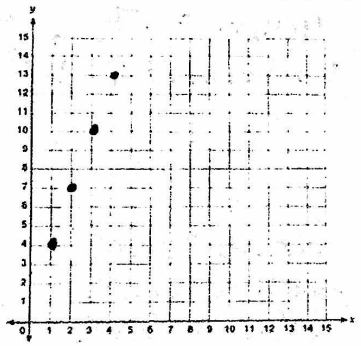
$a_2 = a_1 + 3(1)$

$a_3 = a_1 + 3(2)$

$a_4 = a_1 + 3(3)$

$a_n = a_1 + 3(n-1)$

Arithmetic Explicit Rule: $a_n = a_1 + d(n-1)$



Write an explicit rule for the given sequence. Then answer any additional questions. Assume $n \geq 1$.

5. 5, 12, 19, 26,...

$a_1 = 5 \quad d = 7$

$a_n = 5 + 7(n-1)$

$= 5 + 7n - 7$

$a_n = 7n - 2$

6. Find an explicit formula for a_n for the arithmetic sequence with the following terms: $a_3 = 19$ and $a_5 = 27$.

(3, 19) (5, 27) $d = \frac{27-19}{5-3} = 4$

$a_n = a_1 + d(n-1)$

$a_n = a_1 + 4(n-1)$

$19 = a_1 + 4(3-1)$

$19 - 8 = a_1$

$11 = a_1$

$a_n = 11 + 4(n-1)$

$a_n = 4n + 7$

Arithmetic Series

Find the sum of: 40+37+34+31+28+25+22

The **sum** of a finite arithmetic sequence with n terms (n^{th} partial sum) can be found by:

$S_n = \frac{n}{2}(a_1 + a_n)$ where n = **# of terms** a_1 = **first term** and a_n = **last term**

Find the sum of the finite arithmetic sequence.

Sum of integers from 1 to 35.

$1 + 2 + 3 + \cdots + 35$

$d = 1$

$a_1 = 1$

$a_{35} = 35$

$S_{35} = \frac{35}{2}(1 + 35)$

$S_{35} = 630$

Sum of odd integers from 1 to 57

$1 + 3 + 5 + \cdots + 57$

$n = 29 = \frac{57-1}{2} + 1 \quad d = 2$

$a_1 = 1$

$a_{29} = 57$

$S_{29} = \frac{29}{2}(1 + 57)$

$= 841$

50th partial sum of the arithmetic sequence $-6, -2, 2, 6, \ldots$ $d = 4$

$a_n = a_1 + d(n-1)$

$a_{50} = -6 + 4(50-1)$

$a_{50} = 190$

$S_{50} = \frac{50}{2}(-6 + 190)$

$S_{50} = 4600$

Determine the seating capacity of an auditorium with 30 rows of seats if there are 20 seats in the first row, 22 in the second, 24 in the third row, and so on.

$20 + 22 + 24 + \cdots$

$d = 2 \quad n = 30$

$a_{30} = a_1 + 2(n-1)$

$a_{30} = 20 + 2(29) = 78$

$S_{30} = \frac{30}{2}(20 + 78)$

$S_{30} = 1470$

$\sum_{n=1}^{100}(2 + 3n)$ 100 terms

$a_1 = 2 + 3(1) = 5$

$a_{100} = 2 + 3(100) = 302$

$S_{100} = \frac{100}{2}(5 + 302)$

$S_{100} = 15350$

$\sum_{n=23}^{100}(2 + 3n)$